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FAILURE DISTRIBUTIONS OF SHOCK MODELS

by

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0. Non-Technical Summary

In this paper, a single device shock model is studied. The model we consider consists of a single device which experiences shocks which come from the outside environment. An example is a sensitive electrical component which occasionally experiences a large electrical surge due to a malfunction in the electrical system. Each of these shocks can render the device inoperable, and will at least make it more likely to fail when the next shock arrives.

It is often important to classify the distribution of the time to failure for some item. What we attempt to do in this paper is to examine the one-device shock model and to consider the conditions on the shock process and on the ability of the device to withstand shocks which allow us to do such classifying. We try to find conditions on the process which allow us to place the time-to-failure distribution of the model into one of the common reliability theory classifications.

1. Introduction and Summary

In this paper, we shall study a simple one-device shock model and attempt to establish various qualitative features of its time-to-failure distribution.

Let L be the time of failure of the device and let $\bar{H}(t) = P(L > t)$, and $H(t) = 1 - \bar{H}(t)$. We suppose that the device is subject to a series of shocks whose arrival process is a renewal process with interarrival distribution F. We further suppose that the device survives the first k shocks with probability P_k. We assume that $1 = P_0 \geq P_1 \geq P_2 \geq \cdots$. Then $\bar{H}(t) = \sum_{k=0}^{\infty} P_k[F^{(k)}(t) - F^{(k+1)}(t))]$, where $F^{(k)}$ is the kth convolution of F.

The general approach of this paper is to make various assumptions about the $\{P_k\}$ sequence and the sequence of functions $\{F^{(n)}(t)-F^{(n+1)}(t)\}$ or f, the density of F, and to then try to classify $H(t)$, the time to failure distribution. The classifications common in reliability theory include Increasing Failure Rate (IFR) distributions, Increasing Failure Rate on the Average (IFRA) distributions and distributions with Polya-Frequency of order 2 (PF_2) densities.

The general setup of this paper largely follows that of [1] and [3]. In [3], the shock renewal process was taken to be Poisson, that is, F was assumed to be exponential; and in [1], the shock process was assumed to be non-homogeneous Poisson. In addition, we will apply some of our results to the stochastic wear process introduced by Morey in [6]. A stochastic wear process is an increasing, non-negative stochastic process, where $F(t,x) = P(Z_t \leq x)$ is Totally Positive of order 2 (see Section 3). We will now discuss some of the results of [3] and of our results. If the reader is unfamiliar with some of the terms, he can refer to Sections 2 and 3.

Below, we note the major results of [3], which can be extended, in some natural fashion, to the case of a more general renewal process. Many of the other results, which deal with distribution classifications such as New Better than Used (NBU) and New Better than Used in Expectation (NBUE), seem to be valid only for Poisson renewal processes.

Suppose that $\bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} (\lambda t)^k/k!\, P_k$ where $1 = P_0 \geq P_1 \geq P_2 \geq \cdots$, then

i) If $\theta_k = P_k/P_{k-1}$ is decreasing in $k = 1, 2, \ldots$, i.e., if $\{P_k, k \geq 0\}$ is a PF_2 sequence, then H is IFR.

ii) If $P_k^{1/k}$ is decreasing in $k = 1, 2, \ldots$, then H is IFRA.

iii) If $p_k = P_{k-1} - P_k$ is PF_2, then the density of H is PF_2.

We will extend each of the above results to a more general renewal process.

For i), we assume that $F^{(n)}(t) - F^{(n+1)}(t)$ is TP_3, for ii), we assume that $f^{(n)}(t)$ is TP_2, and in iii), we assume that $f^{(n)}(t)$ is TP_3.

For i) and ii), no further assumptions are made on the $\{P_k\}$ sequence, but for iii), we assume that $p_k = P_k - P_{k-1}$ is decreasing.

In i), ii) and iii), we assume that $\sum_{n=0}^{\infty} (F^{(n)}(t) - F^{(n+1)}(t))\xi^n$ is respectively log concave for $\xi \in [0,1]$, IFRA for $\xi \in [0,1]$ and has a log concave density for $\xi \in [0,1]$.

In each case we get the same result as in [3] and we also show that the assumptions on the quantity, $\sum_{n=0}^{\infty} (F^{(n)}(t) - F^{(n+1)}(t))\xi^n$ are necessary. We also find a large class of densities which satisfy the various assumptions.

For the stochastic wear process, we study the hitting time of Z_t to some random barrier X. We find necessary conditions on the process Z_t and on X so that the distribution of T_X, where $T_X = \inf\{t > 0 | Z_t > X\}$, is either IFR, IFRA or has a PF_2 density

Suppose Z_t is a stochastic wear process with $E[\xi^{Z_t}]$ log concave in t for $\xi \in [0,1]$. Then if $F(t,x) = P(Z_t \leq x)$ is TP_3 and X has a PF_2 density, or if $f(t,x)$, the density of $F(t,x)$, is TP_3 and right continuous in t and X has an IFR distribution, we show that T_X has an IFR distribution.

Also, if Z_t is strictly increasing and $f(t,x)$ is TP_3 and right continuous and X has a PF_2, decreasing density, we show that T_X has a PF_2 density. In all of the above cases, we show the necessity of the condition that $E[\xi^{Z_t}]$ be log concave.

In addition, if $E[\xi^{Z_t}]$ is IFRA for $\xi \in [0,1]$, if $F(t,x)$ is TP_2 and X has a PF_2 density, we show that T_X is IFRA.

2. Distribution Classes in Reliability Theory

In this section, we define some of the classes of distributions used in reliability theory. Throughout, F and H will be distributions on $\mathbb{R}^+$ with $F(0) = 0$. f and h will be respectively the density of F and H (when the distributions are absolutely continuous). $\bar{F}$ and $\bar{H}$ are, respectively, $1-F$ and $1-H$. $F^{(n)}$ is the nth fold convolution of F and $f^{(n)}$ the nth fold convolution of f. $\{P_k\}$ will be a decreasing sequence of non-negative numbers with $P_0 = 1$.

When appropriate, we will have

$$\bar{H}(t) = \sum_{n=0}^{\infty} (F^{(n)}(t) - F^{(n+1)}(t))\, P_n \quad .$$

F is said to be IFR if $\bar{F}(x+t)/\bar{F}(t)$ is decreasing* (when defined) in $t > 0$ for all $x > 0$.

If $\bar{F}(t)^{1/t}$ is decreasing in $t > 0$, F is said to be IFRA. It is well known that F IFR implies that F is IFRA.

F is discrete IFRA if the support of F is the non-negative integers and $\bar{F}(n)^{1/n}$ is decreasing in n (n an integer).

F is discrete IFR if the support of F is the non-negative integers and $\{\bar{F}(n)\}$ is PF_2 (see next section) on the integers. A sequence $\{a_n\}$, $n \geq 0$ is discrete PF_2 if and only if a_{n+1}/a_n is decreasing in n.

If F is absolutely continuous, F IFR is equivalent to $r(t) = f(t)/\bar{F}(t)$ increasing in $t > 0$ ($\bar{F}(t) > 0$) for some version of f.

3. Summary of Total Positivity

Consider any function $K : X \times Y \to \mathbb{R}$, where $X, Y \subset \mathbb{R}$. Define the function $K_{[P]}(\bar{x},\bar{y}) = \det\|K(x_i, y_j)\|_{\substack{i=1,\ldots,P \\ j=1,\ldots,P}}$ on the set $\Delta P(X) \times \Delta P(Y)$ where $\Delta P(X)$ is the set of all P-tuples $(X_1,\ldots,X_P)$ for which $X_i < X_{i+1}$, $i = 1, \ldots, P-1$ and $X_i \in X$. Define $\Delta P(Y)$ similarly.

*Throughout, increasing (decreasing) means non-decreasing (non-increasing).

Definition. $K(x,y)$ is (strictly) totally positive of order r TP_r (STP_r) if $K_{[P]}(\bar{x},\bar{y}) \geq 0$ (> 0), $P \leq r$, $\forall(\bar{x},\bar{y}) \in \triangle P(X) \times \triangle P(Y)$. If $K(x,y)$ is TP_r (STP_r) of all orders, it is said to be TP_∞ (STP_∞).

Definition. If $K(x,y)$ is TP_r and admits a representation $K(x,y) = \phi(y-x)$, we say that ϕ is a Polya Frequency Function of order r (PF_r).

A function is PF_2 if and only if it is log concave. F is IFR if and only if $\bar{F}$ is PF_2. If F has a density f which is PF_2, then F is IFR.

We now note some other characterizations of PF_2 functions and IFRA distributions. From [2], F is IFRA if and only if for each $\lambda > 0$, $\bar{F}(t) - e^{-\lambda t}$ has at most one change of sign, and if one change of sign actually occurs, it occurs from $+$ to $-$.

From [3], $f \geq 0$ is PF_2 on $\mathbb{R}^+$ if and only if for all a, θ, $f(t) - ae^{-\theta t}$ changes sign at most twice, and if two sign changes occur, they are in the order $- + -$. That is, $f(t) - ae^{-\theta t}$ goes from negative to positive to negative.

Examples:

$$\begin{aligned} K(x,y) &= e^{xy} && \text{is } STP_\infty, \\ K(x,y) &= 1_{y<x} && \text{is } TP_\infty. \end{aligned} \qquad (1)$$

We now introduce the notion of variation diminishing. Let $F(t)$ be defined on I where I is an ordered set of the real line. For our purposes, I will always be $\mathbb{R}^+$.

Let

$$S^-(F) = \sup S^-[F(t_1), \ldots, F(t_m)] ,$$

where the sup is taken over $t_1 < t_2 < \cdots < t_m$, $t_i \in I$, m arbitrary and $S^-[x_1, \ldots, x_k]$ is the number of sign changes of the sequence, zeros being ignored.

Let $K(x,y)$ defined on $X \times Y$ be Borel measurable. Assume that $\int_Y K(x,y)\, du(y)$ exists for every x in X. u is taken to be a sigma-finite regular measure on Y. Let f be bounded and Borel measurable on Y and let

$$g(x) = \int_Y K(x,y)\; f(y)\; u(dy) \quad .$$

The proofs of the following three theorems can be found in [4].

<u>Theorem 1</u>. If K is TP_r and the above conditions are fulfilled, then $S^-(g) \leq S^-(f)$ provided that $S^-(f) \leq r-1$.

Also, if f is piecewise continuous, then f and g exhibit the same sequence of signs when their respective arguments traverse the domain of definition from left to right.

<u>Theorem 2</u>. If $K(x,y)$ is TP_r on $(0,\infty] \times (0,\infty]$ and integrable over y with respect to a sigma-finite measure μ on $[0,\infty]$, then the iterates, $L(n,y)$ are TP_r for $n = 1, 2, 3, \ldots, y \geq 0$, where

$$L(1,y) = K(0,y)$$

$$L(n,y) = \int_0^\infty L(n-1, \xi)\, K(\xi,y)\, d\mu(\xi)\,, \qquad n = 2,3,4, \ldots$$

<u>Theorem 3</u>. If $K(\xi,Y)$ is TP_r on $X \times Y$ and $L(\tau,n)$ is TP_m on $Y \times Z$, where X, Y and Z are subsets of the real line and σ is a sigma finite measure on $\mathbb{R}$, then

$$M(\xi,n) = \int_Y K(\xi,\tau)\, L(\tau,n)\, d\sigma(\tau)\,, \qquad \xi \in X,\ n \in Z$$

is $TP_{\min(m,r)}$ assuming the integrals converge absolutely.

<u>Lemma 1</u>. If f is a PF_2 density concentrated on $[0,\infty]$, and if F is the associated distribution and $F^{(n)}$ the nth fold convolution of F, then $F^{(n)}(t) - F^{(n+1)}(t)$ is TP_2. $f^{(n)}(t)$ is also TP_2.

<u>Proof</u>. Apply Theorem 2 with $K(x,y) = f(y-x)$, $d\mu(\xi) = d\xi$ to see that $f^{(n)}(t)$ is TP_2.

Apply Theorem 3 with $K(\xi,\tau) = f(\tau-\xi)$, $L(\tau,m) = 1_{\tau \geq m}$ and $d\sigma(\tau) = d\tau$ to see that $1 - F(t)$ is PF_2.

Finally, note that the product of two TP_2 functions is TP_2 and then apply Theorem 3 with $K(\xi,\tau) = (1 - F(\xi-\tau))1_{\xi \geq \tau}$ and $L(\tau,n) = f^{(n)}(\tau)$ and $d\sigma(\tau) = d\tau$.

4. The Shock Renewal Process

Let us more formally consider the shock renewal process. In continuous time, let N_t be the number of shocks which have arrived by time t, and if we assume that shocks only come at integer times, let N_n be the number of shocks which have arrived by time n.

Then for $\xi \in (0,1)$,

$$E[\xi^{N_t}] = \sum_{n=0}^{\infty} \xi^n P(N_t = n)$$

$$= \sum_{n=0}^{\infty} \xi^n (F^{(n)}(t) - F^{(n+1)}(t))$$

$$= 1 + \frac{\xi-1}{\xi} \sum_{n=1}^{\infty} F^{(n)}(t)\, \xi^n ,$$

$E[\xi^{N_0}] = 1$, and

$$\lim_{t\to\infty} E[\xi^{N_t}] = 1 + \frac{\xi-1}{\xi} \lim_{t\to\infty} \sum_{n=1}^{\infty} F^{(n)}(t)\, \xi^n$$

$$= 1 + \frac{\xi-1}{\xi} \sum_{n=1}^{\infty} \lim_{t\to\infty} F^{(n)}(t)\, \xi^n \qquad \text{(by bounded convergence)}$$

$$= 1 + \frac{\xi-1}{\xi} \sum_{n=1}^{\infty} \xi^n .$$

For $\xi = 0$, let $E[\xi^{N_t}] = P(N_t = 0)$. Note that $E[\xi^{N_t}]$ is continuous in ξ. Our major assumption through the remainder of the paper is that $E[\xi^{N_t}]$ ($E[\xi^{N_n}]$) is either log concave (PF_2) or IFRA (discrete IFRA).

We list some examples below:

(1) If $F(t) = 1 - e^{-\lambda t}$, then

$$E[\xi^{N_t}] = \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{-\lambda t}}{n!} \xi^n = e^{-\lambda t(1-\xi)} .$$

(2) Let $F(t)$ be a gamma distribution with rate 1 and parameter 2. Let f be the density of F. Let $\rho^2 = \xi$,

$$E[\xi^{N_t}] = 1 + \frac{\rho^2-1}{\rho^2} \sum_{n=1}^{\infty} F^{(n)}(t)\, \rho^{2n} .$$

Taking Laplace transforms with

$$g(s) = \int_0^{\infty} e^{-st} E[\xi^{N_t}]\, dt$$

$$g(s) = \frac{1}{s} + \frac{\rho^2-1}{\rho^2} \frac{1}{s} \int_0^{\infty} e^{-st} \sum_{n=1}^{\infty} f^{(n)}(t)\, \rho^{2n}\, dt$$

$$= \frac{1}{s} + \frac{\rho^2-1}{\rho^2} \frac{1}{s} \sum_{n=1}^{\infty} \left(\int_0^{\infty} e^{-st} f(t)\, \rho^2\, dt\right)^n$$

$$= \frac{1}{s} + \frac{\rho^2-1}{\rho^2} \frac{1}{s} \cdot \frac{\frac{\rho^2}{(1+s)^2}}{1 - \frac{\rho^2}{(1+s)^2}}$$

$$= \frac{1}{s}\left[1 + (\rho^2-1) \frac{1}{(1+s)^2 - \rho^2}\right]$$

$$= \frac{1}{s}\left[1 + (\rho^2-1) \cdot \left[\frac{-\frac{1}{2\rho}}{(1+s)+\rho} + \frac{\frac{1}{2\rho}}{(1+s)-\rho}\right]\right] .$$

Reinverting,

$$E[\xi^{N_t}] = \frac{1}{2\rho}\left[(1+\rho)\, e^{-(1-\rho)t} + (\rho-1)\, e^{-(1+\rho)t}\right] .$$

A simple calculation will show that this function is log concave.

(3) Consider a renewal process on the non-negative integers with interval distribution F, where $F(n) = 1 - p^n$, $0 < p < 1$, $q = 1-p$. Such a process is Markovian. It can be shown that $N_n \overset{d}{=} \sum_{k=0}^{n} S_k$ where the $\{S_k\}$ are independent, identically distributed with $P(S_k = 0) = q$, $P(S_k = 1) = p$. Therefore,

$$E[\xi^{N_n}] = \sum_{k=0}^{n} p^k q^{n-k} \xi^k \binom{n}{k}$$

$$= q^n (1 + p\xi/q)^n .$$

Clearly, $E[\xi^{N_n}]$ is a PF_2 function on the non-negative integers.

(4) Let $G(n) = F^{(2)}(n)$, where F is as in (3). If N_n is the discrete renewal process associated with G, then

$$N_n = \sum_{k=0}^{[n/2]} S_k ,$$

where $\{S_k\}$ is as in (3). Letting $\rho = \xi^2$, we have

$$E[\xi^{N_n}] = \sum_{k=0}^{[n/2]} \binom{n}{2k} (p\rho)^{2k} q^{(n-2k)}$$

$$+ \frac{1}{\rho} \sum_{k=0}^{[n/2]-1} \binom{n}{2k+1} (p\rho)^{2k+1} q^{n-(2k+1)}$$

$$= \frac{(p\rho+q)^n + (p\rho-q)^n}{2} \cdot (1 + \frac{1}{\rho}) \ .$$

This sequence is PF_2.

Section 7 contains a discussion on how to construct distributions for which $E[\xi^{N_t}]$ is log concave.

Theorem 4. Let F have a density f which is PF_2. Then H is IFRA for all discrete IFRA sequences $\{P_k\}$ if and only if $E[\xi^{N_t}]^{1/t}$ is decreasing in t for all $\xi \in [0,1]$.

Also, $F^{(n)}(t) - F^{(n+1)}(t)$ TP_2 can replace the condition that f be PF_2.

Theorem 5. Let F be a distribution function whose support is the positive integers and suppose that $f(n) = F(n) - F(n-1)$ is PF_2. Then $H(n)$ is discrete IFRA for all discrete IFRA sequences $\{P_k\}$ if and only if $E[\xi^{N_n}]^{1/n}$ is decreasing in n for all $\xi \in [0,1]$.

Theorem 6. Suppose that $F^{(n)}(t) - F^{(n+1)}(t)$ is TP_3. Then $H(t)$ is IFR for all PF_2 sequences $\{P_k\}$ if and only if $E[\xi^{N_t}]$ is log concave (PF_2) for all $\xi \in [0,1]$.

Theorem 7. Let F be a distribution function whose support is the positive integers and suppose that $F^{(k)}(n) - F^{(k+1)}(n)$ is TP_3. Then $H(n)$ is discrete IFR *for all* PF_2 *sequences* $\{P_k\}$ *if and only if* $E[\xi^{N_n}]$ is PF_2 for all $\xi \in [0,1]$.

Theorem 8. Let $p_k = P_k - P_{k-1}$, $k \geq 1$. Suppose that $f^{(n)}(t)$ is TP_3. Then, $H(t)$ has a PF_2 density for all PF_2, decreasing sequences $\{p_k\}$, if and only if $g(t,\xi) = \sum_{n=1}^{\infty} f^{(n)}(t)\, \xi^n$ is log concave for all $\xi \in [0,1]$.

Before we prove the above theorems, we must first prove a couple of technical lemmas. Lemma 2 is needed for the proofs of Theorems 6 and 7 and Lemma 3 for Theorem 8.

First, we introduce the function,

$$g(\xi, t_1, t_2) = E[\xi^{N_{t_2}}]/E[\xi^{N_{t_1}}] \qquad \text{for } \xi \in (0,1],\ t_2 \geq t_1 \geq 0 .$$

Let $k_t = \inf\{n \geq 0 : P(N_{t_1} = n) > 0\}$. We define

$$\begin{aligned} g(0, t_1, t_2) &= \lim_{\xi \downarrow 0} g(\xi, t_1, t_2) \\ &= P(N_{t_2} = k_{t_1})/P(N_{t_1} = k_{t_1}) . \end{aligned}$$

Lemma 2. If $F^{(n)}(t) - F^{(n+1)}(t) = P(N_t = n)$ is TP_2, then for any $t_2 \geq t_1 \geq 0$ for which $\bar{H}(t_1) > 0$, there exists a $\xi \in [0,1]$ for which $g(\xi, t_1, t_2) = \bar{H}(t_2)/\bar{H}(t_1)$.

Proof. $g(1, t_1, t_2) = 1$ and g is continuous in $\xi \in [0,1]$. So, if we can prove the existence of a $\xi \in [0,1]$ for which

$$0 \le g(\xi, t_1, t_2) \le \bar{H}(t_2)/\bar{H}(t_1) \le 1 = g(1, t_1, t_2) ,$$

then by the continuity of g, the lemma is proved.

Now, $g(0, t_1, t_2) = P(N_{t_2} = k_{t_1})/P(N_{t_1} = k_{t_1})$, so it suffices to show that $g(0, t_1, t_2) \le \bar{H}(t_2)/\bar{H}(t_1)$, that is,

$$P(N_{t_2} = k_{t_1})\, \bar{H}(t_1) \le \bar{H}(t_2)\, P(N_{t_1} = k_{t_1}) ,$$

$$\begin{aligned} P(N_{t_2} = k_{t_1})\, \bar{H}(t_1) &= \sum_{\ell \ge k_{t_1}} P(N_{t_2} = k_{t_1})\, P(N_{t_1} = \ell)\, P_\ell \\ &\le \sum_{\ell \ge k_{t_1}} P(N_{t_1} = k_{t_1})\, P(N_{t_2} = \ell)\, P_\ell \\ &= P(N_{t_1} = k_{t_1})\, \bar{H}(t_2) . \end{aligned}$$

The inequality follows as $P(N_t = k)$ is TP_2.

The following lemma is proven in greater generality than needed for this section. The more general form is needed for Section 8. The reader can skip this proof for the moment and just refer to the corollary.

We will let μ be some σ-finite measure on $\mathbb{R}^+$, g some non-negative, decreasing function on $\mathbb{R}^+$ and $f(t,x)$ some function on $\mathbb{R}^+ \times \mathbb{R}^+$ which is TP_2 on $\mathbb{R}^+ \times A$ where A is the support of μ, and which is right continuous in x for each t.

Lemma 3. For f, g, μ as above, if $\int_0^\infty f(t,x)\,\mu(dx)$ is decreasing in t, then for $t_1 \le t_2$,

$$\overline{\lim_{\xi \downarrow 0}} \left[\frac{\int_0^\infty f(t_2,x)\,\xi^x\,\mu(dx)}{\int_0^\infty f(t_1,x)\,\xi^x\,\mu(dx)} \right] \le \frac{\int_0^\infty f(t_2,x)\,g(x)\,\mu(dx)}{\int_0^\infty f(t_1,x)\,g(x)\,\mu(dx)}$$

$$\le \lim_{\xi \uparrow 1} \left[\frac{\int_0^\infty f(t_2,x)\,\xi^x\,\mu(dx)}{\int_0^\infty f(t_1,x)\,\xi^x\,\mu(dx)} \right] ,$$

where we assume that all denominations are non-zero and all integrals finite.

Proof. Let

$$V \equiv \mu, \quad U(x,y) = f(t_2,x)\,f(t_1,y) - f(t_2,y)\,f(t_1,x) .$$

We will first prove the right-hand side inequality. As

$$\lim_{\xi \uparrow 1} \int_0^\infty f(t,x)\,\xi^x\,\mu(dx) = \int_0^\infty f(t,x)\,\mu(dx) ,$$

we must show that

$$\int_0^\infty f(t_2,x)\,g(x)\,\mu(dx) \int_0^\infty f(t_1,x)\,\mu(dx)$$

$$\le \int_0^\infty f(t_2,x)\,\mu(dx) \int_0^\infty f(t_1,x)\,g(x)\,\mu(dx) .$$

Now,

$$0 \geq \iint_{x > y \geq 0} U(x,y)\ (g(x) - g(y))\ \mu(dx)\ V(dy)$$

$$= \iint_{x > y \geq 0} U(x,y)\ g(x)\ \mu(dx)\ V(dy)$$

$$+ \iint_{y > x \geq 0} U(x,y)\ g(x)\ \mu(dx)\ V(dy)\ .$$

Substituting for $U(x,y)$ and seperating each of the above integrals gives us the desired inequality.

Now, we consider the left-hand side of the inequality. Let

$$k(t) = \inf\{x > 0 \mid \int_0^x f(t,x)\ \mu(ds) > 0\}\ .$$

If $\mu(\{k(t_1)\}) > 0$, the inequality follows immediately from the fact that $f(t,x)$ is TP_2 and $g(x) \geq 0$. Suppose that $\mu(\{k(t_1)\}) = 0$,

$$\overline{\lim_{\xi \downarrow 0}} \left[\frac{\int_0^\infty f(t_2,x)\ \xi^x\ \mu(dx)}{\int_0^\infty f(t_1,x)\ \xi^x\ \mu(dx)}\right] = \overline{\lim_{\xi \downarrow 0}} \left[\frac{\int_{k(t_1)}^\infty f(t_2,x)\ \xi^{x-k(t_1)}\ \mu(dx)}{\int_{k(t_1)}^\infty f(t_1,x)\ \xi^{x-k(t_1)}\ \mu(dx)}\right]$$

$$\leq \overline{\lim_{\epsilon \downarrow 0}}\ \overline{\lim_{\xi \downarrow 0}} \left[\frac{\int_{k(t_1)}^{k(t_1)+\epsilon} f(t_2,x)\ \xi^{x-k(t_1)}\ \mu(dx)}{\int_{k(t_1)}^{k(t_1)+\epsilon} f(t_1,x)\ \xi^{x-k(t_1)}\ \mu(dx)}\right]$$

$$\leq \lim_{\epsilon \downarrow 0} \frac{f(t_2,\ k(t_1) + \epsilon)}{f(t_1,\ k(t_1) + \epsilon)}\ .$$

As f is TP_2,

$$\frac{f(t_2,k(t_1)+\epsilon)}{f(t_1,k(t_1)+\epsilon)} \leq \frac{\int_{k(t_1)+\epsilon} f(t_2,x)\ g(x)\ \mu(dx)}{\int_{k(t_1)+\epsilon} f(t_1,x)\ g(x)\ \mu(dx)} .$$

Letting $\epsilon \to 0$ on both sides, and noting the above string of inequalities shows that

$$\overline{\lim_{\xi \downarrow 0}} \frac{\int_0^\infty f(t_2,x)\ \xi^x\ \mu(dx)}{\int_0^\infty f(t_1,x)\ \xi^x\ \mu(dx)} \leq \frac{\int_0^\infty f(t_2,x)\ g(x)\ \mu(dx)}{\int_0^\infty f(t_1,x)\ g(x)\ \mu(dx)}$$

as required.

We will now prove the needed corollary. Let p_k be a decreasing non-negative sequence. Let $f(t)$ be the density of some non-negative random variable.

Corollary. If $f^{(n)}(t)$ is TP_2, then for any $t_1 < t_2$, $\{p_k\}$ decreasing, there exists a $\xi \in [0,1]$ such that

$$\frac{h(t_2)}{h(t_1)} = \frac{\sum_{k=1}^\infty f^{(k)}(t_2)\ p_k}{\sum_{k=1}^\infty f^{(k)}(t_1)\ p_k} = \frac{\sum_{k=1}^\infty f^{(k)}(t_2)\ \xi^k}{\sum_{k=1}^\infty f^{(k)}(t_1)\ \xi^k} .$$

Proof. We can apply Lemma 3 with $g(k) = p_k$, $f(t,n) = f^{(n)}(t)$ and μ counting measure on the positive integers. We can extend g to $\mathbb{R}^+$ and f to $\mathbb{R}^+ \times \mathbb{R}^+$ in an appropriate manner. By Lemma 3,

$$\lim_{\xi \downarrow 0} \frac{\sum_{k=1}^{\infty} f^{(k)}(t_2)\, \xi^k}{\sum_{k=1}^{\infty} f^{(k)}(t_1)\, \xi^k} \le \frac{h(t_2)}{h(t_1)} \le \lim_{\xi \uparrow 1} \frac{\sum_{k=1}^{\infty} f^{(k)}(t_2)\, \xi^k}{\sum_{k=1}^{\infty} f^{(k)}(t_1)\, \xi^k} .$$

The corollary follows by the continuity of $\sum_{k=1}^{\infty} f^{(k)}(t)\, \xi^k$ in ξ.

Proof of Theorem 4.

$$\bar{H}(t) = \sum_{n=0}^{\infty} [F^{(n)}(t) - F^{(n+1)}(t)]\, P_n$$

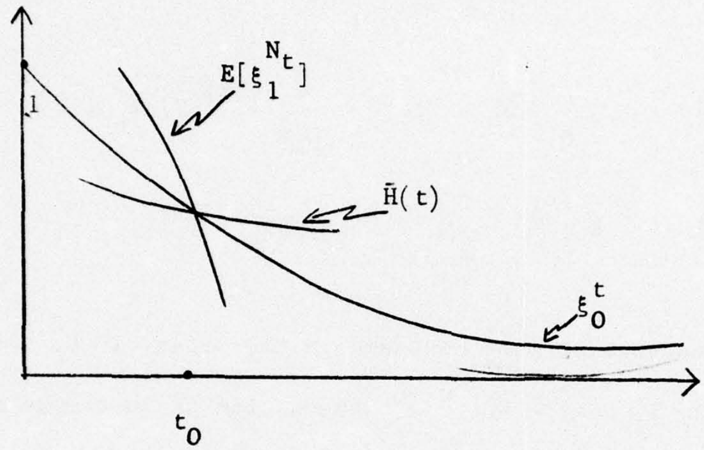
$$\bar{H}(t) - E[\xi^{N_t}] = \sum_{n=0}^{\infty} [F^{(n)}(t) - F^{(n+1)}(t)]\, (P_n - \xi^n) .$$

As $P_n^{1/n}$ decreases by assumption, the sign of $P_n - \xi^n$ changes sign at most once, and if so, from + to -. As f is PF_2, $F^{(n)}(t) - F^{(n+1)}(t)$ is TP_2 by Lemma 1. So, by Theorem 1, $\bar{H}(t) - E[\xi^{N_t}]$ has at most one sign change and if it does change sign, it is from + to -.

We suppose that $H(t) = 1 - \bar{H}(t)$ is not IFRA. It then follows from [2], p. 89, that there exists a $0 < \xi_0 < 1$ and a t_0 for which $\bar{H}(t)$ crosses ξ_0^t from below at $t = t_0$. Now

$$\lim_{\xi \downarrow 0} E[\xi^{N_{t_0}}] = P(N_{t_0} = 0) \le H(t_0) .$$

As $E[\xi^{N_t}]$ is continuous in ξ and $E[1^{N_t}] = 1 \geq \bar{H}(t)$, it follows that there exists a $\xi_1 \in [0,1]$ for which $E[\xi_1^{N_{t_0}}] = \bar{H}(t_0) = \xi_0^{t_0}$.



As $E[\xi_1^{N_t}]^{1/t}$ is decreasing $(1 - E[\xi_1^{N_t}]$ is IFRA), $E[\xi_1^{N_t}]$ has a downcrossing with respect to ξ_0^t at $t = t_0$.

Therefore, $\bar{H}(t)$ crosses $E[\xi_1^{N_t}]$ from below at $t = t_0$. So the sign of $\bar{H}(t) - E[\xi_1^{N_t}]$ goes from - to +. This is a contradiction, so $\bar{H}(t)$ must be IFRA.

$\Rightarrow$ We are given then $H(t)$ is IFRA for all sequences $\{P_k\}$ where $1 = P_0 \geq P_1 \geq \cdots$, $P_k^{1/k}$ decreasing. Let $P_k = \xi^k$, $0 \leq \xi \leq 1$,

$$\bar{H}(t) = \sum_{n=0}^{\infty} (F^{(n)}(t) - F^{(n+1)}(t))\, \xi^n = E[\xi^{N_t}] \quad .$$

So $1 - E[\xi^{N_t}]$ is IFRA $(E[\xi^{N_t}]^{1/t}$ decreasing) as H is IFRA.

Proof of Theorem 5.

$\Leftarrow$ $f(n) = F(n+1) - F(n)$ is PF_2 by assumption. By Lemma 1, $f^{(k)}(n)$ is TP_2,

$$\bar{H}(n) = \sum_{k=0}^{\infty} P(N_n = k)\, P_k = \sum_{k=0}^{\infty} f^{(k)}(n)\, P_k ,$$

$$\bar{H}(n) - E[\xi^{N_n}] = \sum_{k=0}^{\infty} f^{(k)}(n)\, (P_k - \xi^k) .$$

$P_k - \xi^k$ changes sign at most once and in the order $+$ to $-$. Therefore, by Theorem 1, $\bar{H}(n) - E[\xi^{N_n}]$ changes sign at most once and in the order $+$ to $-$.

Suppose $\bar{H}(n)$ is not discrete IFRA. Then we can find a $\xi_0 \in [0,1)$ and an $n_0 > 0$ for which $\bar{H}(n_0) < \xi_0^{n_0}$ and $\bar{H}(n_0+1) > \xi_0^{n_0+1}$. We can, as in the previous proof, assert that there exists a $\xi_1 \in [0,1]$ for which $\xi_0^{n_0} > E[\xi_1^{N_{n_0}}] > \bar{H}(n_0)$.

As $E[\xi_1^{N_{n_0}}] < \xi_0^{n_0}$ and $E[\xi_1^{N_n}]^{1/n}$ is decreasing, $E[\xi_1^{N_{n_0+1}}] < \xi_0^{n_0+1}$, $E[\xi_1^{N_n}]$ goes below $\bar{H}(n)$ at $n = n_0+1$. That is, the sign of $\bar{H}(n) - E[\xi_1^{N_n}]$ changes from $+$ to $-$ as n goes from n_0 to n_0+1.

$\Rightarrow$ Let $P_n = \xi^n$ and proceed as in Theorem 5.

Proof of Theorem 6.

$$\bar{H}(t) = \sum_{n=0}^{\infty} [F^{(n)}(t) - F^{(n+1)}(t)]\, P_n$$

$$\bar{H}(t) - aE[\xi^{N_t}] = \sum_{n=0}^{\infty} [F^{(n)}(t) - F^{(n+1)}(t)]\, (P_n - a\xi^n) .$$

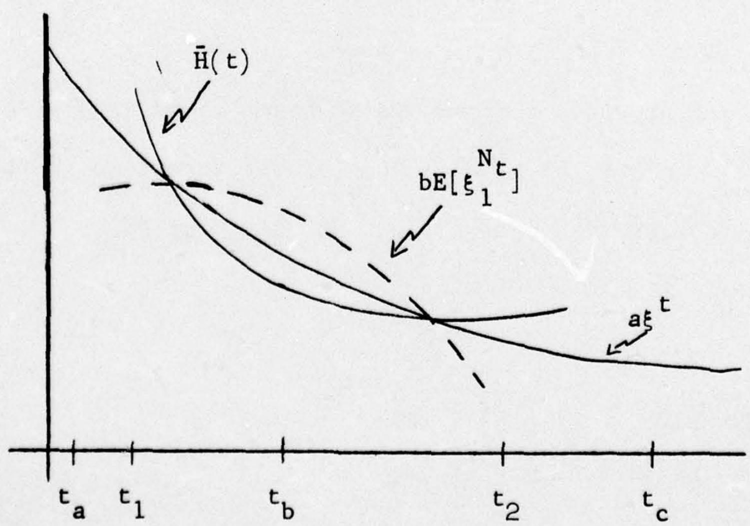
As $\{P_n\}$ is PF_2, P_{n+1}/P_n is decreasing, so the sign of $P_n - a\xi^n$ changes at most twice and in the order $- + -$. As $F^{(n)}(t) - F^{(n+1)}(t)$

is TP_3, we can assert by Theorem 1 that the sign of $\bar{H}(t) - aE[\xi^{N_t}]$ changes sign at most twice and in the order $- + -$.

We know that H is IFR if and only if for any a and ξ, the number of sign changes of $\bar{H}(t) - a\xi^t$ number at most two, and if there are two, they are in the order $- + -$.

Let us suppose the contrary. That is, assume that there exists an a, ξ for which the sign of $\bar{H}(t) - a\xi^t$ goes from $-$ to $+$ to $-$. Refer to the below diagram. By Lemma 2, we can find a b, ξ_1 with $bE[\xi_1^{N_{t_1}}] = a\xi^{t_1} = \bar{H}(t_1)$, $bE[\xi_1^{N_{t_2}}] = a\xi^{t_2} = \bar{H}(t_2)$. As $bE[\xi^{N_t}]$ is log concave by assumption, we have that for $t_1 < t < t_2$, $bE[\xi_1^{N_t}] \geq a\xi^t$. It is clear that there exists t_a, t_b, t_c, $t_a < t_1 < t_b < t_2 < t_c$ for which $\bar{H}(t_a) - bE[\xi_1^{N_{t_a}}] > 0$, $\bar{H}(t_b) < bE[\xi_1^{N_{t_b}}]$, and $\bar{H}(t_c) > bE[\xi_1^{N_{t_c}}]$. This, however, is in contradiction to the initial result about the sign change pattern of $H(t) - aE[\xi^{N_t}]$ for all a, ξ. So H is IFR.

$\Rightarrow$ Again, let $P_n = \xi^n$ and the converse is immediate.



Proof of Theorem 7.

$\Leftarrow$ A review of Lemma 2 shows that it is still applicable for discrete time,

$$\bar{H}(n) = \sum_{k=0}^{\infty} [F^{(k)}(n) - F^{(k+1)}(n)] P_k = \sum_{k=0}^{\infty} f^{(k)}(n) P_k ,$$

$$\bar{H}(n) - aE[\xi^{N_n}] = \sum_{k=0}^{\infty} f^{(k)}(n) (P_k - a\xi^k) .$$

As $P_k - a\xi^k$ changes sign at most twice, and in the order $- + -$, and as $f^{(k)}(n)$ is TP_3, then by Theorem 1, $\bar{H}(n) - aE[\xi^{N_n}]$ changes sign at most twice and in the order $- + -$.

We suppose that $\bar{H}(n)$ is not PF_2, or equivalently that there exists constants $a', \xi_0', n_1, n_2, n_3$, where $n_1 < n_2 < n_3$ are integers for which $\bar{H}(n_1) > a', \xi_0'^{n_1}$, $\bar{H}(n_2) < a' \xi_0'^{n2}$, $\bar{H}(n_3) > a' \xi_0'^{n_3}$.

Making use of the fact that any two exponential curves cross at most once, we can find a new pair of constants, a and ξ_0, for which $\bar{H}(n_1) = a\xi_0^{n_1}$, $\bar{H}(n_2) < a\xi_0^{n_2}$, $\bar{H}(n_3) = a\xi_0^{n_3}$.

As in the previous theorem, using Lemma 2, we find a b, ξ_1, with $bE[\xi_1^{N_{n_1}}] = \bar{H}(n_1)$, $bE[\xi_1^{N_{n_3}}] = \bar{H}(n_3)$. We proceed as in the previous theorem to get a contradiction. So $\bar{H}(n)$ is PF_2.

$\Rightarrow$ Let $P_n = \xi^n$.

Proof of Theorem 8.

$\Leftarrow$ Let h be the density of H,

$$h(t) = \sum_{n=1}^{\infty} f^{(n)}(t)\, p_n \quad ,$$

$$h(t) - ag(t,\xi) = \sum_{n=1}^{\infty} f^{(n)}(t)\,(P_n - a\xi^n) \quad .$$

We now have the same setup as in Theorem 6. We now may use the Corollary to Lemma 3 in the identical way that we used Lemma 2 in the proof of Theorem 6.

$\Rightarrow$ Let $p_k = \frac{1-\xi}{\xi} \cdot \xi^k$.

7. Densities for Which $E[\xi^{N_t}]$ is log Concave

As was mentioned earlier, the question of finding distributions with the properties required in Theorems 4 through 8 is formidable. Below, we identify a class of densities for which $E[\xi^{N_t}]$ is log concave.

Definition. Let $\mathcal{E}_1$ be the set of functions Q on $[0,\infty]$ with the representation $Q(s) = (e^{\delta s} s^k \prod_{n=1}^{\infty} (1 + \lambda_i s)$, $\lambda_i \geq 0$, $\sum_{i=1}^{\infty} \lambda_i < \infty$, $\delta \geq 0$, k a non-negative integer and $Q(0) = C$.

Theorem 9. $h(u)$ is a PF_∞ function on $[0,\infty]$ if the reciprocal of the Laplace transform of h is in $\mathcal{E}_1$.

Proof. The proof of this theorem can be found in [4].

Note that

$$Q(s) = \prod_1^n (\lambda_i + s) / \prod_1^n \lambda_i$$

$$= \prod_1^n (1 + 1/\lambda_i \, s)$$

is the reciprocal of the Laplace transform of the convolution of n exponential densities with rates λ_i.

Let $G(t,\xi) = 1 - E[\xi^{N_t}]$.

<u>Theorem 10</u>. Let F have a density $f \cdot G(t,\xi)$ has a PF_∞ density $q(t,\xi)$, for $\xi \in [0,1)$ if the Laplace transform of f has a reciprocal of the form $\prod_1^n (\lambda_i + s)/\prod_1^n \lambda_i$ and $\left(\prod_{i=1}^n (\lambda_i + s) - \prod_{i=1}^n \lambda_i\right)$ has all real roots.

<u>Proof</u>.

$$q(t,\xi) = \frac{1-\xi}{\xi} \sum_{n=1}^{\infty} \xi^n f^{(n)}(t) \quad .$$

Let

$$q_\alpha(s,\xi) = \int_0^\infty e^{-st} q(t,\xi)\, dt$$

$$f_\alpha(s) = \int_0^\infty e^{-st} f(t)\, dt \quad .$$

Then,

$$q_\alpha(s,\xi) = \frac{1-\xi}{\xi} \frac{f_\alpha(s)\xi}{1 - f_\alpha(s)\xi} \quad .$$

We want to show that $q(t,\xi)$ is a PF_∞ density, and as it is clearly a density, it suffices by Theorem 9 to show that $q_\alpha^{-1}(s,\xi)$ is in $\mathcal{E}_1$ for all $\xi \in [0,1)$,

$$q_\alpha^{-1}(s,\xi) = \frac{1}{1-\xi}\,(q_\alpha^{-1}(s) - \xi)$$

$$= \frac{1}{1-\xi}\,(\Pi(\lambda_i+s) - \xi \prod_{i=1}^{n} \lambda_i)\,(\sum_{i=1}^{n} \lambda_i)^{-1} .$$

Now, $q_\alpha^{-1}(s,\xi)$ is in $\mathcal{E}_1$ if $\phi_\xi(s) = \Pi(\lambda_i+s) - \xi \prod_{i=1}^{n} \lambda_i$ has all real, non-positive roots. For $\xi \in [0,1]$, $\phi_\xi(s) > 0$ for $s > 0$. So $\phi_\xi(s)$ has no positive roots. An inspection of the polynomial ϕ_ξ will show that ϕ_ξ has all real roots for $\xi \in [0,1]$ if and only if ϕ_1 has all real roots.

<u>Corollary</u>. Let f be the convolution of two exponential densities. Then $q(t,\xi)$ is PF_∞ for $\xi \in [0,1)$.

<u>Proof</u>. $q_\alpha^{-1}(s,\xi) = \frac{1}{1-\xi}\,((\lambda_1+s)\,(\lambda_2+s) - \xi\,\lambda_1\,\lambda_2))(\lambda_1\lambda_2)^{-1}$.

We need to see if $\phi_1(s) = (\lambda_1+s)\,(\lambda_2+s) - \lambda_1\lambda_2$ has all real roots $\phi_1(s) = s(s + \lambda_1 + \lambda_2)$, so the conditions of Theorem 10 are satisfied.

<u>Note</u>: The result in Theorem 10 only holds for $\xi \in [0,1)$, but it can easily be extended to the case of $\xi = 1$ by continuity or by noting that $q(t,\xi) \equiv 0$ for $\xi = 1$.

8. Stochastic Wear Processes

In [6], Morey introduces a class of non-decreasing stochastic processes, which he called stochastic wear processes (SWP). $\{Z_t, t \geq 0\}$ is a SWP if $P(Z_t \leq x)$ is TP_2 in t and x, Z_t is increasing and Z_t is non-negative.

Morey shows that if Z_t is a SWP with stationary independent increments, then the random variable T_x defined as $\inf\{t > 0 | Z_t > x\}$ is an IFR random variable. He further asserts that for any non-negative random variable X, T_X is also IFR. The latter result is not true. Simply take Z_t to be identically t, note that $P(Z_t \leq x) = 1_{\{t \leq x\}}$ which is TP_2 and therefore Z_t is a SWP with stationary, independent increments. Also, $P(T_X \leq t) = P(Z_t \geq X) = P(X \leq t)$, so T_X IFR implies that X is IFR, which is clearly not true in general.

From now on, we will be considering a non-decreasing stochastic process Z_t with $Z_0 = 0$, $F(t,x) = P(Z_t \leq x)$ and $f(t,x)$ the density of F for each t (if it exists).

We will prove the analogues of Theorems 4 through 8 for processes which can take values on all of $\mathbb{R}^+$.

First, we need the following lemma which is similar to Lemmas 2 and 3.

<u>Lemma 4</u>. Suppose that $F(t,x)$ is TP_2, $q \geq 0$ and $\phi(t) = \int_{x=0}^{\infty} F(t,x)\, g(x)dx$. Then, for all $t_1 < t_2$,

$$\lim_{\xi \downarrow 0} \left[\frac{\int_{x=0}^{\infty} F(t_2,x)\, \xi^x\, dx}{\int_{x=0}^{\infty} F(t_1,x)\, \xi^x\, dx}\right] \leq \frac{\phi(t_2)}{\phi(t_1)} \leq \lim_{\xi \uparrow 1} \left[\frac{\int_{x=0}^{\infty} F(t_2,x)\, \xi^x\, dx}{\int_{x=0}^{\infty} F(t_1,x)\, \xi^x\, dx}\right] .$$

Proof. The left-hand side of the inequality follows by Lemma 3. Now, $\phi(t_2) \le \phi(t_1)$ as $F(t_2,x) \le F(t_1,x)$. Also

$$\int_{x=0}^{\infty} F(t,x)\,\xi^x\,dx = \int_{x=0}^{\infty}\int_{u=0}^{\infty} f(t,u)\,\xi^x\,du\,dx$$

$$= \int_{u=0}^{\infty}\int_{x=u}^{\infty} f(t,u)\,\xi^u\,\xi^{x-u}\,du\,dx$$

$$= \int_{u=0}^{\infty}\int_{y=0}^{\infty} f(u,t)\,\xi^u\,\xi^y\,dy\,du$$

$$= -1/\ln \xi \; E[\xi^{Z_t}] \,.$$

So,

$$\lim_{\xi \uparrow 1}\left[\frac{\int_{x=0}^{\infty} F(t_2,x)\,\xi^x\,dx}{\int_{x=0}^{\infty} F(t_1,x)\,\xi^x\,dx}\right] = \lim_{\xi \uparrow 1} \frac{-1/\ln \xi \; E[\xi^{Z_{t2}}]}{-1/\ln \xi \; E[\xi^{Z_{t1}}]} = 1 \,.$$

Let $G(x) = P(X \le x)$, $\bar{G}(x) = 1 - G(x)$. Let $g(x)$ be the density of G. Let $\bar{H}(t) = P(T > t)$, $H(t) = 1 - H(t)$. Let $h(t)$ be the density of $H(t)$ (if it exists). Then,

i) $\bar{H}(t) = P(T > t) = \int_{x=0}^{\infty} F(t,x)\,g(x)\,dx.$

ii) $\bar{H}(t) = P(T > t) = \int_{x=0}^{\infty} f(t,x)\,\bar{G}(x)\,dx.$

If Z_t is strictly increasing in t,

iii) $h(t) = \int_{x=0}^{\infty} f(t,x)\,g(x)\,dx.$

Theorem 11. If $F(t,x)$ is TP_3, then $\bar{H}(t)$ is PF_2 (H is IF) for all g PF_2 if and only if $E[\xi^{Z_t}] = -\int_{x=0}^{\infty} f(t,x)\, \xi^x\, dx\, (\ln \xi)$ is log concave in t for all $\xi \in [0,1]$.

Proof.

$$\bar{H}(t) - a\int_{x=0}^{\infty} F(t,x)\, \xi^x\, dx = \int_{x=0}^{\infty} F(t,x)\, (g(x) - a\xi^x)\, dx \quad .$$

$\bar{H}(t) - a\int_{x=0}^{\infty} F(t,x)\, \xi^x\, dx$ changes sign at most twice and in the order $- + -$ as $g(x)$ is PF_2 and $F(t,x)$ is TP_3 (Theorem 1). Now, apply Lemma 4 and argue as in Theorem 5.

$\Rightarrow$ Let $g(x) = \xi^x$.

Note that $E[\xi^{Z_t}]$ is log concave if and only if $\int_{x=0}^{\infty} F(t,x)\, \xi^x\, dx$ is log concave.

We state the next three theorems without proof. They each depend either on Lemma 3 or Lemma 4.

Theorem 12. Let $f(t,x)$ be right continuous for each t and TP_3. Then, $\bar{H}(t)$ is log concave (H is IFR) for all $\bar{G}(t)$ PF_2 if and only if $E[\xi^{Z_t}]$ is log concave for all $\xi \in [0,1]$.

Theorem 13. Suppose that Z_t is strictly increasing. Let $f(t,x)$ be right continuous for each t and TP_3. Then, $h(t)$ is PF_2 for all $g(t)$ PF_2, and decreasing, if and only if $E[\xi^{Z_t}]$ is log concave for all $\xi \in [0,1]$.

Theorem 14. Let $f(t,x)$ be right continuous and TP_2. Then $H(t)$ is IFRA for all $G(t)$ IFRA if and only if $E[\xi^{Z_t}]^{1/t}$ is decreasing.

Example. Let N_t be a non-homogenous Poisson Process with $E(N_t) = \Lambda(t)$. Consider the compound renewal process whose arrival process is N_t and whose jump size distribution is M with $M(0) = 0$. Assume that M has a density m. Let T_n be the size of the nth jump. Let

$$Z_t = \begin{cases} \sum_{k=1}^{N_t} T_k, & \text{if } N_t \geq 1 \\ 0, & \text{if } N_t = 0 \end{cases}$$

Then

$$P(Z_t \leq x) = \sum_{n=0}^{\infty} e^{-n(t)} \frac{n(t)^k}{k!} M^{(n)}(x) \quad .$$

Let $F(t,x) = P(Z_t \leq x)$. Let $f(t,x)$ be the density of F on $(0,\infty]$. We have

$$f(t,x) = \sum_{n=1}^{\infty} e^{-n(t)} \frac{n(t)^k}{k!} m^{(n)}(x) \,, \qquad x > 0 \,.$$

Suppose $m(x)$ is PF_3. Then $m^{(n)}(x)$ is TP_3. Since $e^{-n(t)} \frac{n(t)^k}{k!}$ is TP_∞, we know by Theorem 3 that $f(t,x)$ is TP_3 on $t \geq 0$, $x > 0$. Extend $f(t,x)$ to $t \geq 0$, $x \geq 0$ so that it is still TP_3. Now,

$$E[\xi^{Z_t}] = e^{-\Lambda(t)(1-m_\xi)} \qquad \text{where} \qquad m_\xi = \int_{x=0}^{\infty} m(x)\, \xi^x \, dx \,.$$

So $E[\xi^{Z_t}]$ is log concave if $\Lambda(t)$ is convex.

Therefore, by Theorem 12, if Λ is convex and m PF_3, the hitting time distribution of Z_t to any random barrier with an IFR distribution is IFR.

9. Markov Processes

The questions we have been asking about hitting time distributions have very simple answers if the process is Markovian and spatially homogeneous. All of the below is taken directly from [5].

Henceforth, Z_t is a Markov Process, which is spatially homogeneous. We let $\phi_t(s,x)$ be the density of $P(Z_{t+s} \leq x | Z_t = 0)$.

Theorem 15. If $\phi_t(s,x)$ is TP_2 in s and x for each t and if $f(x)$ is PF_2, then $c(t) = \int_0^\infty \phi_0(t,x)\ f(x)\ dx$ is PF_2.

Proof.

$$c(t+s) = \int_0^\infty \phi_0(t+s,\ x)\ f(x)\ dx$$

$$= \int_0^\infty \int_0^x \phi_0(t,\xi)\ \phi_t(s,\ x-\xi)\ f(x)\ dx$$

$$= \int_0^\infty \phi_0(t,\xi) \int_0^\infty \phi_t(s,u)\ f(u+\xi)\ du\ d\xi\ .$$

As f is PF_2, $f(u+\xi)$ is RR_2 (see [4]). By a slight extension of Theorem 3 (see [4]), $\int_0^\infty \phi_t(s,u)\ f(u+\xi)\ du$ is RR_2, so as $\phi_0(t,\xi)$ is TP_2, by Theorem 3, $c(t+s)$ is RR_2. But $c(t+s)$ RR_2 implies that

$c(t)$ is PF_2. Using Theorem 15, we can get the same results as in Section 8 for temporally homogeneous Markov Processes with fewer and weaker assumptions. Note that for appropriate choices of f, c can be interpreted as the hitting time distribution or density of the Markov process to a random barrier.

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In this paper, we considers a single device shock model. The device experiences shocks from the environment, each of which can render the device inoperable. We find sets of conditions ~~on~~ of the shock process and ~~on~~ the ability of the device to survive shocks are found so that the time to failure distribution of the device falls into one of the common reliability theory classifications. We extend these results are extended to the case where the shock process can be viewed as continuous.

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